Open Problem:

Polynomial linearly-convergent method for geodesically convex optimization?

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Chris Criscitiello — Institute of Mathematics, EPFL

David Martínez-Rubio — Zuse Institute Berlin and Technische Universität

Nicolas Boumal — Institute of Mathematics, EPFL

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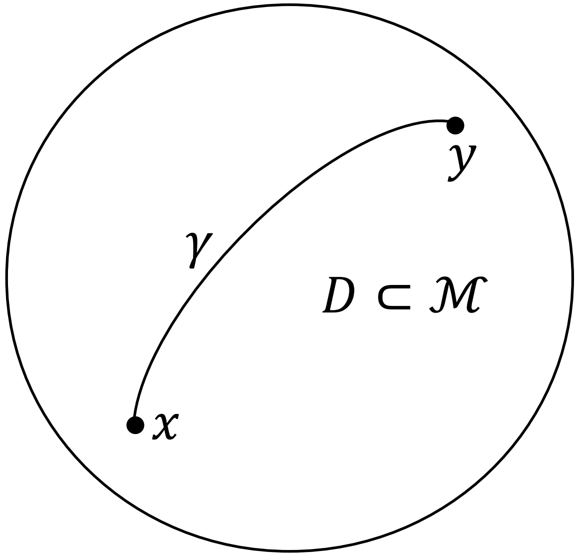

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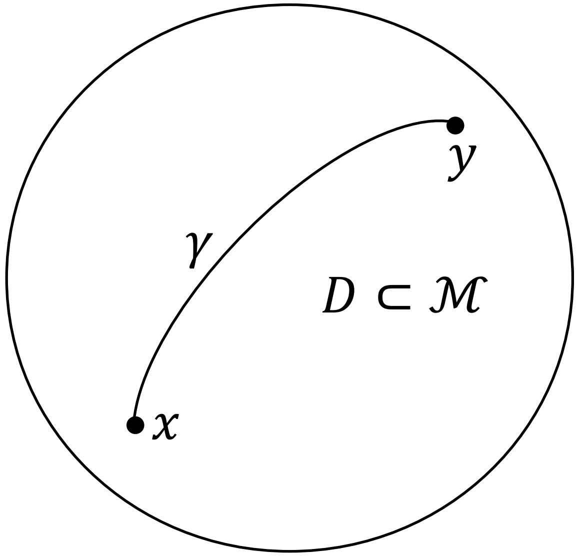

$f : D \to \mathbb{R}$ is M-Lipschitz:

- $|f(x) - f(y)| \leq M \,\mathrm{dist}(x, y)$, for all $x, y \in D$

Ellipsoid method

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Full convexity not needed.
Just need halfspace at each iteration.

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Q: Is there a first-order deterministic algorithm with the following properties?

- (Linear convergence) uses at most $O(\text{poly}(d, \zeta) \log(\epsilon^{-1}))$ subgradient queries to find a point with target accuracy ϵ, and
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$\zeta \sim 1 + r\sqrt{K}$

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- (Linear convergence) uses at most $O(\text{poly}(d, \zeta) \log(\epsilon^{-1}))$ subgradient queries to find a point with target accuracy ϵ, i.e., $f(x) - f^* \leq \epsilon \cdot Mr$, and
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Our contribution:

- Partial solution: the case of constant curvature (hyperbolic spaces, hemispheres)
- Get others interested in it!

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Note: Such algorithms exist under additional assumptions (strong g-convexity, smoothness, 2nd-order robustness) [Zhang & Sra'16, Allen-Zhu et al.'18, Hirai et al.'23]

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Operator scaling (Gurvits'04, Burgisser'19, ...)

- robust covariance estimation
- matrix normal models
- variant on polynomial identity testing, etc.

Pos def matrices with affine-invariant metric, $\mathcal{M} = SL(n)/SO(n)$

Constant curvature

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Algo

Define $R = \frac{1}{\sqrt{K}}$, $\quad x_0 = x_{\text{ref}}$,

For $k = 0, \ldots, T = \lceil \zeta \log(\epsilon^{-1}) \rceil$

$$x_{k+1} = \frac{\epsilon}{4}\text{-approx solution of subproblem } \min_{\bar{B}(x_k, R) \cap D} f$$

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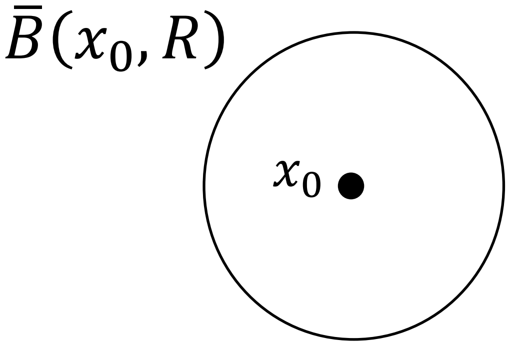

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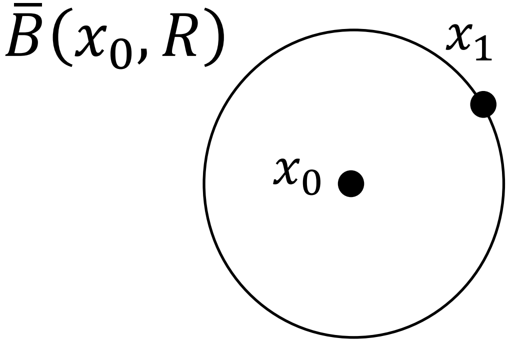

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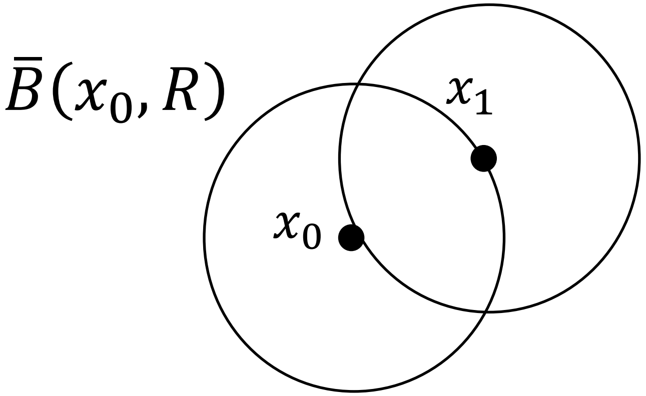

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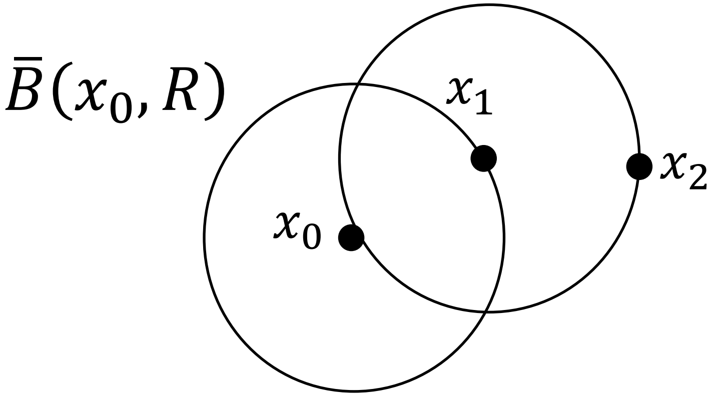

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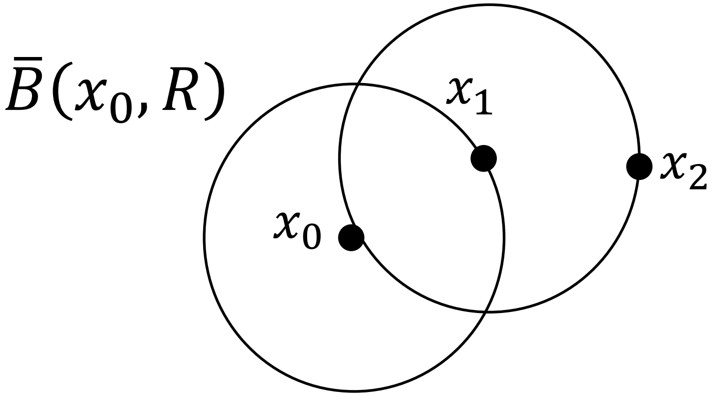

- MR'23, Theorem 7
- Works for any Riemannian Manifold!

Constant curvature

$$\min_{x \in \bar{B}(x_k, R) \cap D} f(x), \text{ with } R = \frac{1}{\sqrt{K}}$$

Goal: $f(x_{k+1}) - \min_{\bar{B}(x_k, R) \cap D} f \leq \frac{\epsilon}{4} \cdot MR$

Assume $\mathcal{M} = \mathbb{H}^d$ is a hyperbolic space (constant curvature -1)

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Key tool: Geodesic map with base point x_k is a diffeomorphism

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with $\phi(x_k) = 0$ and which maps halfspaces in $\mathbb{H}^d$ to halfspaces in $B(0,1) \subset \mathbb{R}^d$

$$\{x \in \mathbb{H}^d : \langle g, \log_y(x) \rangle \leq 0\} \leftrightarrow \{\tilde{x} \in B(0,1) : \tilde{g}^\top(\tilde{x} - \tilde{y}) \leq 0\}$$

$$\tilde{x} = \phi_k(x), \tilde{y} = \phi_k(y)$$
$$\tilde{g} = d\phi_k(x)[g]$$

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Geodesics map of $\mathbb{H}^d$ given by Beltrami Klein model (*explicit formula*)

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Pull back Riemannian problem to Euclidean space via geodesic map

$$\min_{\tilde{x} \in B(0,\tilde{R}) \cap \phi_k^{-1}(D)} \left(f \circ \phi_k^{-1}\right)(\tilde{x})$$

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Takes $O(d^2 \log(\epsilon^{-1}))$ queries, each requiring $O(d^2)$ arithmetic operations.

How to go beyond constant curvature?

Beltrami's theorem: The only Riemannian manifolds which admit geodesic maps to Euclidean space are those of constant curvature.

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Maybe replace geodesic maps with exponential map and use comparison theorems? Not clear …

Other possible approaches

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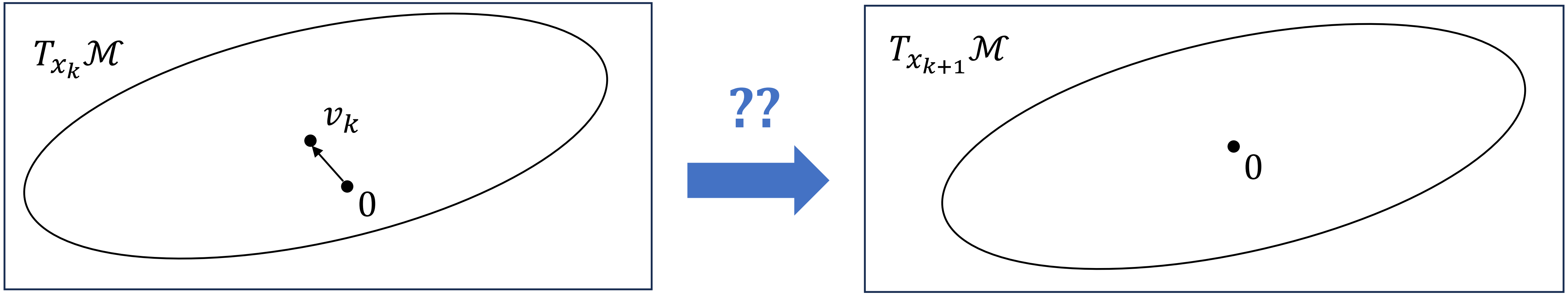

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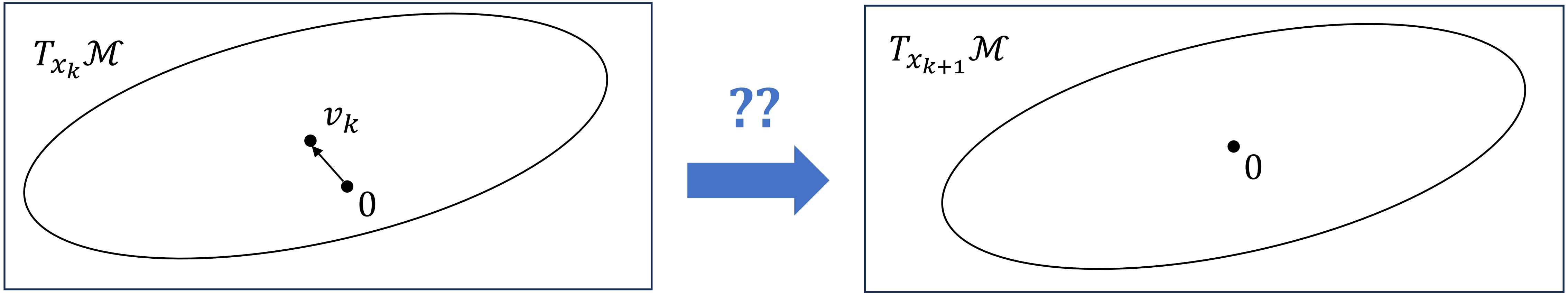

Kim and Yang'22 introduced a way of transferring balls between tangent spaces.

It is not clear how to generalize their results to ellipsoids.

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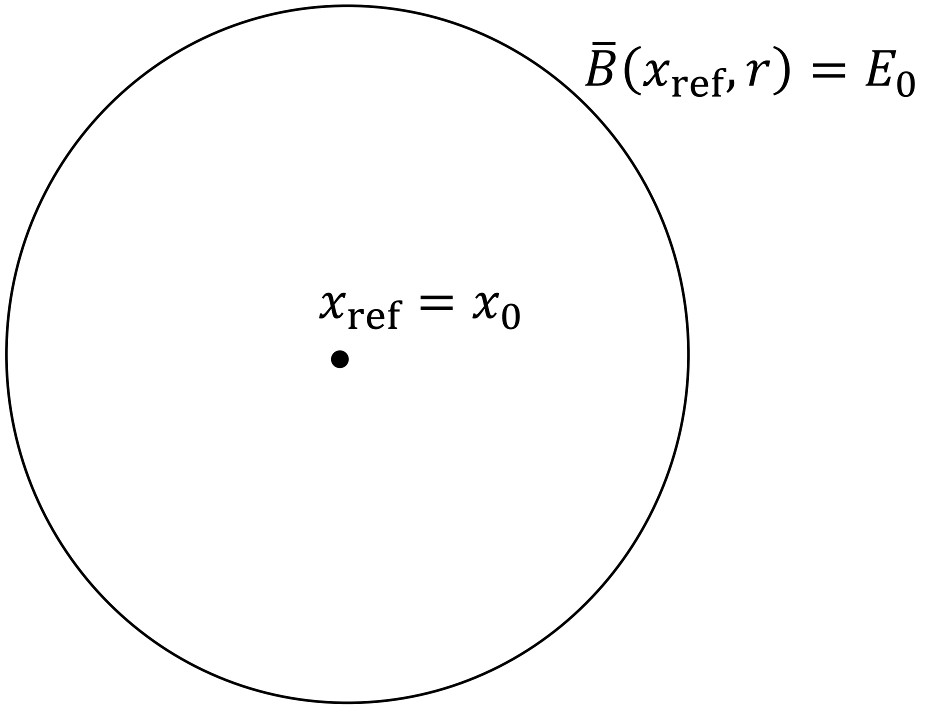

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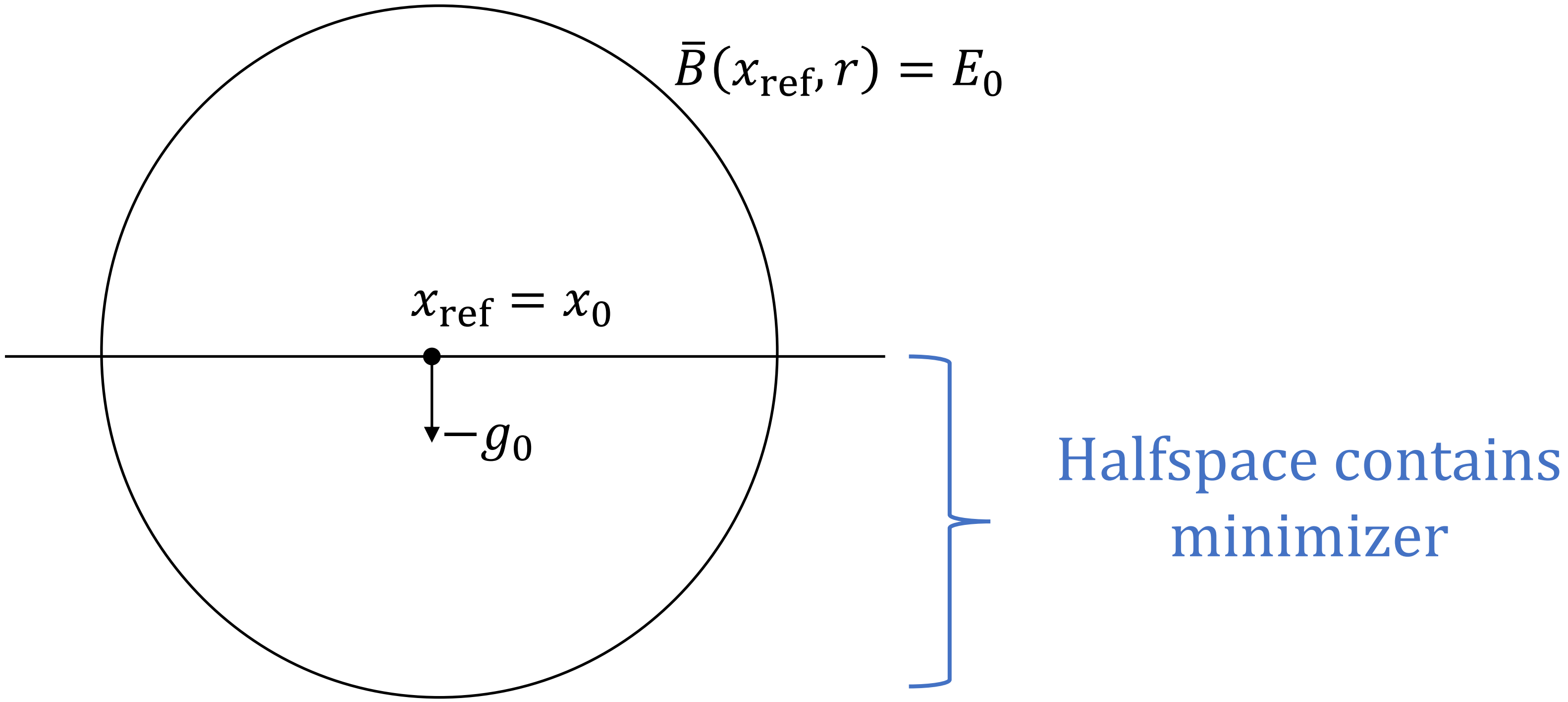

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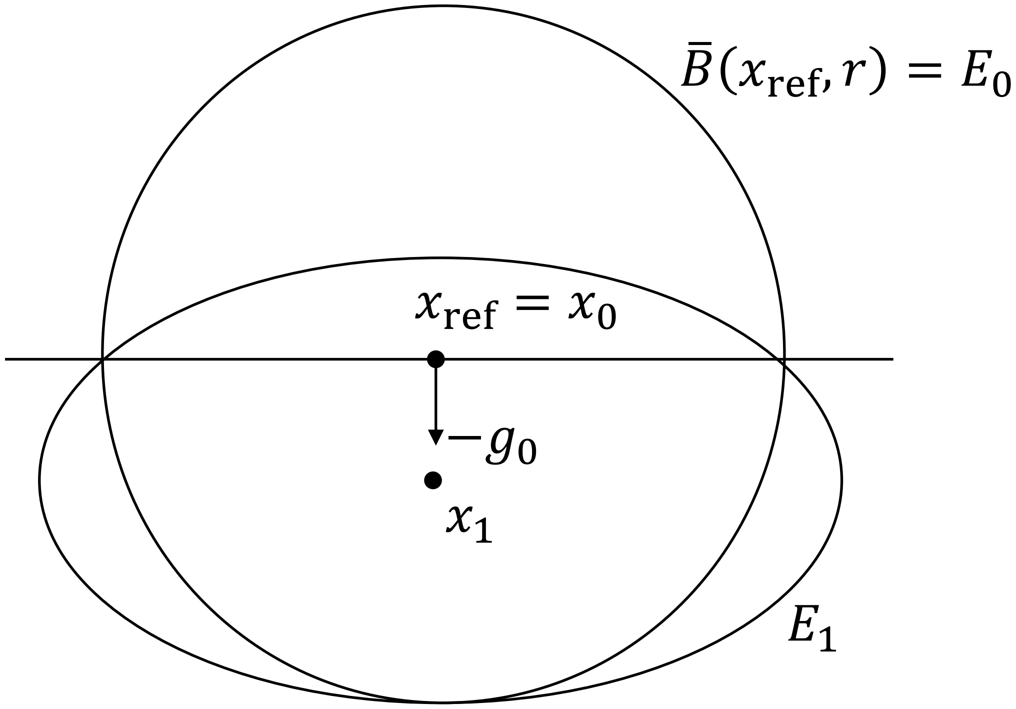

E_1 is minimum-volume ellipsoid containing intersection of halfspace and ball E_0

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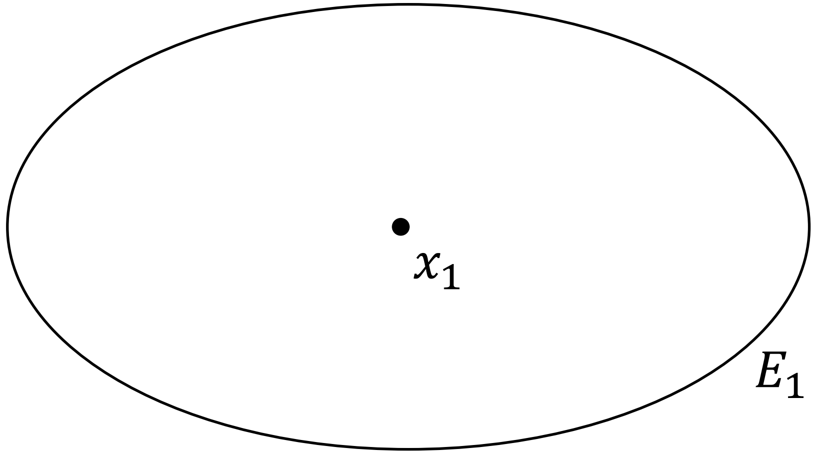

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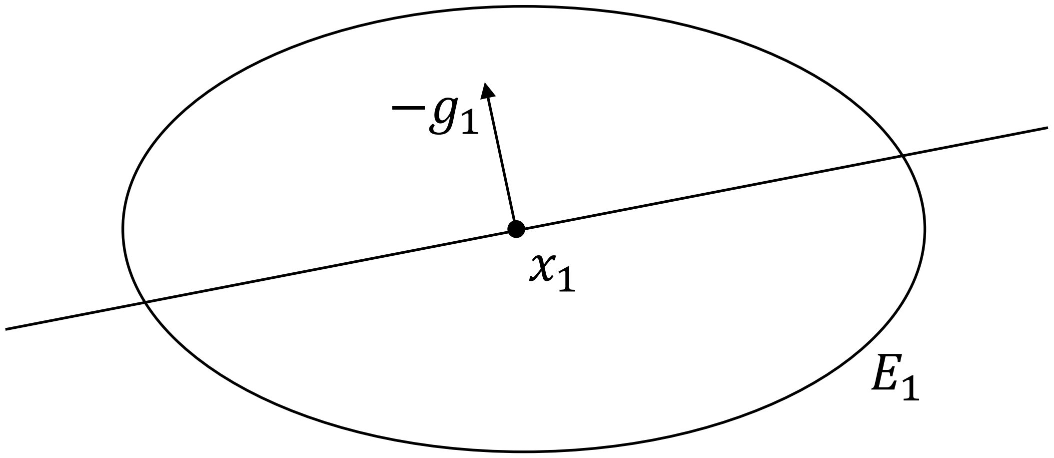

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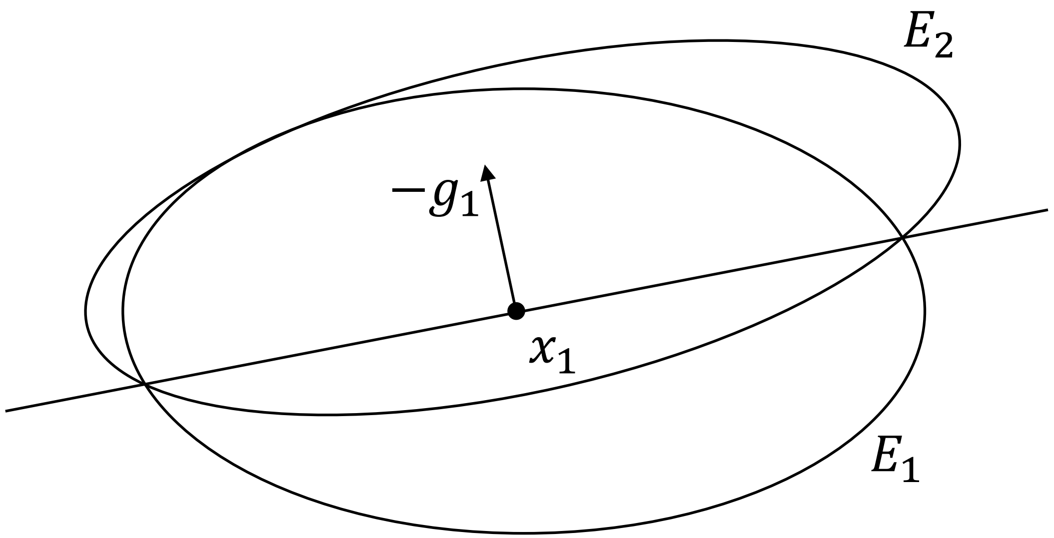

E_2 is minimum-volume ellipsoid containing intersection of halfspace and E_1